

EXAMPLES OF WEAKLY AMENABLE DISCRETE QUANTUM GROUPS

AMAURY FRESLON

ABSTRACT. We prove that the free orthogonal and free unitary quantum groups $\mathbb{FO}^+(F)$ and $\mathbb{FU}^+(F)$ are weakly amenable and that their Cowling-Haagerup constant is equal to 1 for any $F \in GL_n(\mathbb{C})$ such that $F\overline{F} \in \mathbb{R} \cdot \text{Id}$. This is achieved by estimating the completely bounded norm of the projections on the coefficients of their irreducible representations and then by proving the Haagerup property for these non-necessarily unimodular discrete quantum groups.

1. INTRODUCTION

The *free orthogonal* and *free unitary* quantum groups were constructed by A. Van Daele and S. Wang in [30, 33]. They are defined as universal C^* -algebras generalizing the algebras of continuous functions on the classical orthogonal and unitary groups, together with some additional structure turning them into *compact quantum groups*. From then on, these compact quantum groups have been studied from various points of view : probabilistic, geometric and operator algebraic. In particular, their reduced C^* -algebras and von Neumann algebras form interesting classes of examples somehow in the same way as those arising from discrete groups. After the first works of T. Banica [2, 3], it appeared that these operator algebras are closely linked to free group algebras. This link was made more clear by the results of S. Vaes and R. Vergnioux [29] on exactness and factoriality and those of M. Brannan [10] on the Haagerup property and the metric approximation property in the unimodular case.

These works naturally raise the question of weak amenability for free quantum groups. It has been strongly suspected for some time that they have a Cowling-Haagerup constant equal to 1, and this is what we prove in the present paper. To do this, we show that the completely bounded norm of the projections on coefficients of a fixed irreducible representation (i.e. on "words of fixed length") grows polynomially. This fact can then easily be combined with M. Brannan's proof of the Haagerup property to yield weak amenability when $F = \text{Id}$. In the other (non unimodular) cases, we first have to adapt the proof of M. Brannan to

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get nice multipliers implementing the Haagerup property. This can be done using the same averaging principle as in the unimodular setting.

Weak amenability for locally compact groups was originally defined by M. Cowling and U. Haagerup in [14] and studied in the context of real simple Lie groups by J. de Cannière, M. Cowling and U. Haagerup in [14, 17]. In the discrete setting, many examples were provided by N. Ozawa's result [24] stating that all Gromov hyperbolic groups are weakly amenable. Weak amenability has recently attracted a lot of attention since it is a key ingredient in some of S. Popa's deformation/rigidity techniques, see for example [25, 26]. Another feature of this approximation property is that it provides a numerical invariant which carries to the associated operator algebras and may thus give a way to distinguish them. An introduction to approximation properties for classical discrete groups can be found in [13, Chapter 12], though no knowledge on this subject will be required afterwards.

Let us now briefly outline the organization of the paper. In Section 2, we recall some basic facts about compact and discrete quantum groups and we fix notations. We also give some fundamental definitions and results concerning free quantum groups. We then introduce weak amenability for discrete quantum groups in Section 3. Subsection 4.1 contains the first technical part of our result, reducing the problem to controlling the norms of certain blocks of analogs of operator-valued functions on the discrete quantum groups considered. Another technical result is worked out in Subsection 4.2 to obtain a suitable bound on the completely bounded norm of the projection on some fixed irreducible representation. In the final section, we extend the techniques of [10] to prove the Haagerup property for all free orthogonal and unitary quantum groups. We then combine it with the estimate of the preceding section to prove our weak amenability result.

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2. PRELIMINARIES

2.1. Notations. For two Hilbert spaces H and K , $\mathcal{B}(H, K)$ will denote the set of bounded linear maps from H to K and $\mathcal{B}(H) := \mathcal{B}(H, H)$. In the same way we will use the notations $\mathcal{K}(H, K)$ and $\mathcal{K}(H)$ for compact linear maps. We will denote by $\mathcal{B}(H)_*$ the predual of $\mathcal{B}(H)$, i.e. the Banach space of all normal linear forms on $\mathcal{B}(H)$. On any tensor product $H \otimes H'$ of Hilbert spaces, we define the flip operator

$$\Sigma : \begin{cases} H \otimes H' & \rightarrow & H' \otimes H \\ x \otimes y & \mapsto & y \otimes x \end{cases}$$

We will use the usual leg-numbering notations : for an operator X acting on a tensor product, we set $X_{12} := X \otimes 1$, $X_{23} := 1 \otimes X$ and $X_{13} := (\Sigma \otimes 1)(1 \otimes X)(\Sigma \otimes 1)$. The identity map of an algebra A will be denoted ι_A or simply ι if there is no possible confusion. For a subset B of a topological vector space C , $\overline{\text{span}} B$ will denote the *closed linear span* of B in C . The symbol \otimes will denote the *minimal* (or spatial) tensor product of C^* -algebras or the topological tensor product of Hilbert spaces. The spatial tensor product of von Neumann algebras will be denoted $\overline{\otimes}$ and the algebraic tensor product (over \mathbb{C}) will be denoted \odot .

2.2. Compact and discrete quantum groups. Discrete quantum groups will be seen as duals of compact quantum groups in the sense of Woronowicz. We briefly review the basic theory of compact quantum groups as introduced in [37]. Another survey, encompassing the non-separable case, can be found in [23]. Emphasis has been put on the explicit description of the associated L^2 -space since this will prove crucial in the sequel.

Definition 2.1. A *compact quantum group* \mathbb{G} is a pair $(C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital C^* -algebra and $\Delta : C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital $*$ -homomorphism such that

$$\begin{aligned} (\Delta \otimes \iota) \circ \Delta &= (\iota \otimes \Delta) \circ \Delta \\ \overline{\text{span}}\{\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))\} &= C(\mathbb{G}) \otimes C(\mathbb{G}) \\ \overline{\text{span}}\{\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)\} &= C(\mathbb{G}) \otimes C(\mathbb{G}) \end{aligned}$$

The main feature of compact quantum groups is the existence of a Haar state which is both left and right invariant (see [37, Thm 1.3]).

Theorem 2.2 (Woronowicz). *Let \mathbb{G} be a compact quantum group. There is a unique Haar state on \mathbb{G} , that is to say a state h on $C(\mathbb{G})$ such that for all $a \in C(\mathbb{G})$,*

$$\begin{aligned} (\iota \otimes h) \circ \Delta(a) &= h(a).1 \\ (h \otimes \iota) \circ \Delta(a) &= h(a).1 \end{aligned}$$

Let $(L^2(\mathbb{G}), \pi_h, \xi_h)$ be the associated GNS construction and let $C_{\text{red}}(\mathbb{G})$ be the image of $C(\mathbb{G})$ under the GNS map π_h . It is called the *reduced C^* -algebra* of \mathbb{G} . Let W be the unique unitary operator on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ such that

$$W^*(\xi \otimes \pi_h(a)\xi_h) = (\pi_h \otimes \pi_h) \circ \Delta(a)(\xi \otimes \xi_h)$$

for $\xi \in L^2(\mathbb{G})$ and $a \in C(\mathbb{G})$, and let $\widehat{W} := \Sigma W^* \Sigma$. Then W is a *multiplicative unitary* in the sense of [1], i.e. $W_{12}W_{13}W_{23} = W_{23}W_{12}$ and we have the following equalities :

$$C_{\text{red}}(\mathbb{G}) = \overline{\text{span}}\{(\iota \otimes \mathcal{B}(L^2(\mathbb{G}))_*)(W)\} \text{ and } \Delta(x) = W^*(1 \otimes x)W.$$

Moreover, we can define the *dual discrete quantum group* $\widehat{\mathbb{G}} = (C_0(\widehat{\mathbb{G}}), \widehat{\Delta})$ by

$$C_0(\widehat{\mathbb{G}}) = \overline{\text{span}}\{(\mathcal{B}(L^2(\mathbb{G}))_* \otimes \iota)(W)\} \text{ and } \widehat{\Delta}(x) = \Sigma W(x \otimes 1)W^* \Sigma.$$

The two von Neumann algebras associated to these quantum groups are then

$$L^\infty(\mathbb{G}) = C_{\text{red}}(\mathbb{G})'' \text{ and } \ell^\infty(\widehat{\mathbb{G}}) = C_0(\widehat{\mathbb{G}})''$$

where the bicommutants are taken in $\mathcal{B}(L^2(\mathbb{G}))$. The coproducts extend to normal maps on these von Neumann algebras and one can prove that $W \in L^\infty(\mathbb{G}) \overline{\otimes} \ell^\infty(\widehat{\mathbb{G}})$. The Haar state of \mathbb{G} extends to a state on $L^\infty(\mathbb{G})$.

2.3. Irreducible representations and the GNS construction. We will need in the sequel an explicit description of the GNS construction of the Haar state h using the following notion of irreducible representation of a compact quantum group.

Definition 2.3. A *representation* of a compact quantum group \mathbb{G} on a Hilbert space H is an operator $u \in L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(H)$ such that $(\Delta \otimes \iota)(u) = u_{13}u_{23}$. It is said to be *unitary* if the operator u is unitary.

Definition 2.4. Let \mathbb{G} be a compact quantum group and let u and v be two representations of \mathbb{G} on Hilbert spaces H_u and H_v respectively. An *intertwiner* (or *morphism*) between u and v is a map $T \in \mathcal{B}(H_u, H_v)$ such that $v(1 \otimes T) = (1 \otimes T)u$. The set of intertwiners between u and v will be denoted $\text{Mor}(u, v)$.

A representation u will be said to be *irreducible* if $\text{Mor}(u, u) = \mathbb{C} \cdot \text{Id}$ and it will be said to be *contained* in v if there is an isometric intertwiner between u and v . We will say that two representations are *equivalent* (resp. *unitarily equivalent*) if there is an intertwiner between them which is an isomorphism (resp. a unitary). Let us define two fundamental operations on representations.

Definition 2.5. Let \mathbb{G} be a compact quantum group and let u and v be two representations of \mathbb{G} on Hilbert spaces H_u and H_v respectively. The *direct sum* of u and v is the diagonal sum of the operators u and v seen as an element of $L^\infty(\mathbb{G}) \otimes \mathcal{B}(H_u \oplus H_v)$. It is a representation denoted $u \oplus v$. The *tensor product* of u and v is the element $u_{12}v_{13} \in L^\infty(\mathbb{G}) \otimes \mathcal{B}(H_u \otimes H_v)$. It is a representation denoted $u \otimes v$.

The following generalization of the classical Peter-Weyl theorem holds (see [37, Section 6]).

Theorem 2.6 (Woronowicz). *Every representation of a compact quantum group is equivalent to a unitary one. Every irreducible representation of a compact quantum group is finite dimensional and every unitary representation is unitarily equivalent to a sum of irreducible ones. Moreover, the linear span of the coefficients of all irreducible representations is a dense Hopf $*$ -subalgebra of $C(\mathbb{G})$ denoted $\text{Pol}(\mathbb{G})$.*

Let $\text{Irr}(\mathbb{G})$ be the set of isomorphism classes of irreducible unitary representations of \mathbb{G} . If $\alpha \in \text{Irr}(\mathbb{G})$, we will denote by u^α a representative of the class

α and by H_α the finite dimensional Hilbert space on which u^α acts. There are isomorphisms

$$C_0(\widehat{\mathbb{G}}) = \bigoplus_{\alpha \in \text{Irr}(\mathbb{G})} \mathcal{B}(H_\alpha) \text{ and } \ell^\infty(\widehat{\mathbb{G}}) = \prod_{\alpha \in \text{Irr}(\mathbb{G})} \mathcal{B}(H_\alpha).$$

The minimal central projection in $\ell^\infty(\widehat{\mathbb{G}})$ corresponding to the identity of $\mathcal{B}(H_\alpha)$ will be denoted p_α .

We now proceed to describe explicitly the GNS representation of the Haar state using the irreducible representations. For any $\alpha \in \text{Irr}(\mathbb{G})$, there is a unique (up to unitary equivalence) irreducible representation, called the *contragredient representation* of α and denoted $\bar{\alpha}$, such that $\text{Mor}(\varepsilon, \alpha \otimes \bar{\alpha}) \neq \{0\} \neq \text{Mor}(\varepsilon, \bar{\alpha} \otimes \alpha)$, ε denoting the trivial representation. This yields an antilinear isomorphism $j_\alpha : H_\alpha \rightarrow H_{\bar{\alpha}}$. The matrix $j_\alpha^* j_\alpha \in \mathcal{B}(H_\alpha)$ is unique up to multiplication by a real number. We will say that j_α is *normalized* if $\text{Tr}(j_\alpha^* j_\alpha) = \text{Tr}((j_\alpha^* j_\alpha)^{-1})$ (this only determines j_α up to some complex number of modulus one, but this is of no consequence in our context). In that case we will set $Q_\alpha = j_\alpha^* j_\alpha$, $\dim_q(u^\alpha) = \text{Tr}(Q_\alpha) = \text{Tr}(Q_\alpha^{-1})$ and $t_\alpha(1) = \sum j_\alpha(e_i) \otimes e_i$, where (e_i) is some fixed orthonormal basis of H_α . We will also set $u_{i,j}^\alpha = (\iota \otimes e_i^*) u^\alpha (\iota \otimes e_j)$. Note that by construction, $t_\alpha : \mathbb{C} \rightarrow H_{\bar{\alpha}} \otimes H_\alpha$ is a morphism of representations. Let us define a map

$$\Psi_\alpha : \begin{cases} H_{\bar{\alpha}} \otimes H_\alpha & \rightarrow C_{\text{red}}(\mathbb{G}) \\ \eta \otimes \xi & \mapsto \pi_h[(1 \otimes j_{\bar{\alpha}}(\eta)^*) u^\alpha (1 \otimes \xi)] \end{cases}$$

According to [37, Eq. 6.8] we have, for any $z, z' \in H_{\bar{\alpha}} \otimes H_\alpha$,

$$h(\Psi_\alpha(z)^* \Psi_\alpha(z')) = \frac{1}{\dim_q(\alpha)} \langle z, z' \rangle.$$

and $\Psi = \bigoplus_\alpha \sqrt{\dim_q(\alpha)} \Psi_\alpha \cdot \xi_h : \bigoplus_\alpha (H_{\bar{\alpha}} \otimes H_\alpha) \rightarrow L^2(\mathbb{G})$ is an isometric isomorphism of Hilbert spaces. If we let $E_{i,j}$ denote the operator on H_α sending e_i to e_j and the other vectors of the basis to 0, we can define another map

$$\Phi_\alpha : \begin{cases} H_{\bar{\alpha}} \otimes H_\alpha & \longrightarrow \mathcal{B}(H_\alpha) \\ j_\alpha(e_i) \otimes e_j & \mapsto E_{i,j} \end{cases}$$

Now, we observe that $\Theta_\alpha = \Psi_\alpha \circ \Phi_\alpha^{-1} : \mathcal{B}(H_\alpha) \rightarrow C_{\text{red}}(\mathbb{G})$ sends $E_{i,j}$ to $\pi_h(u_{i,j}^\alpha)$ and that

$$\begin{aligned} h(\Theta_\alpha(E_{i,j})^* \Theta_\alpha(E_{k,l})) &= \frac{1}{\dim_q(\alpha)} \langle \Phi_\alpha^{-1}(E_{i,j}), \Phi_\alpha^{-1}(E_{k,l}) \rangle \\ &= \frac{1}{\dim_q(\alpha)} \langle j_\alpha(e_i) \otimes e_j, j_\alpha(e_k) \otimes e_l \rangle \\ &= \frac{\delta_{j,l}}{\dim_q(\alpha)} \langle Q_\alpha e_i, e_k \rangle \\ &= \frac{1}{\dim_q(\alpha)} \text{Tr}(Q_\alpha E_{i,j}^* E_{k,l}). \end{aligned}$$

Thus, if we endow $\mathcal{B}(H_\alpha)$ with the scalar product $\langle A, B \rangle_\alpha = \dim_q(\alpha)^{-1} \text{Tr}(Q_\alpha A^* B)$, we get an isometric isomorphism of Hilbert spaces

$$\Theta = \oplus_\alpha \Theta_\alpha \cdot \xi_h : \oplus_\alpha \mathcal{B}(H_\alpha) \rightarrow L^2(\mathbb{G}).$$

Note that the duality map $S_\alpha : A \mapsto \langle A, \cdot \rangle_\alpha$ being bijective on the finite dimensional space $\mathcal{B}(H_\alpha)$, one can endow $\oplus_\alpha \mathcal{B}(H_\alpha)_*$ with a Hilbert space structure making it isomorphic to $L^2(\mathbb{G})$ via $\Theta \circ (\oplus_\alpha S_\alpha^{-1})$. This is the "natural" isomorphism since it sends $\omega \in \mathcal{B}(H_\alpha)_*$ to $\pi_h[(\iota \otimes \omega)(u^\alpha)] \cdot \xi_h$.

Let u^α and u^β be two irreducible representations of \mathbb{G} and assume, for the sake of simplicity, that *every irreducible subrepresentation of $u^\alpha \otimes u^\beta$ appears with multiplicity one*. This is no restriction in the case of free quantum groups that we will be considering (see Theorem 2.10). Let $v_\gamma^{\alpha, \beta} : H_\gamma \rightarrow H_\alpha \otimes H_\beta$ be an isometric intertwiner. Note that $v_\gamma^{\alpha, \beta} Q_\gamma = (Q_\alpha \otimes Q_\beta) v_\gamma^{\alpha, \beta}$. We have,

$$\begin{aligned} (\iota \otimes \omega_{\xi, \eta})(u^\alpha)(\iota \otimes \omega_{\xi', \eta'})(u^\beta) &= (\iota \otimes \omega_{\xi, \eta} \otimes \omega_{\xi', \eta'})(u_{12}^\alpha u_{13}^\beta) \\ &= (\iota \otimes \omega_{\xi, \eta} \otimes \omega_{\xi', \eta'})(u^\alpha \otimes u^\beta) \\ &= (\iota \otimes \omega_{\xi, \eta} \otimes \omega_{\xi', \eta'})(\sum_{\gamma \subset \alpha \otimes \beta} (\iota \otimes v_\gamma^{\alpha, \beta}) u^\gamma (\iota \otimes v_\gamma^{\alpha, \beta})^*) \\ &= \sum_{\gamma \subset \alpha \otimes \beta} (\iota \otimes [\omega_{\xi, \eta} \otimes \omega_{\xi', \eta'}]^\gamma)(u^\gamma) \end{aligned}$$

where $\omega^\gamma(x) = \omega(v_\gamma^{\alpha, \beta} \circ x \circ (v_\gamma^{\alpha, \beta})^*)$ for $\omega \in \mathcal{B}(H_\alpha \otimes H_\beta)_*$. Using the duality map S_α^{-1} , we can write the map induced on $C_{\text{red}}(\mathbb{G})$ by the product under our identification : for $A \in \mathcal{B}(H_\alpha)$ and $B \in \mathcal{B}(H_\beta)$,

$$\Theta_\alpha(A) \cdot \Theta_\beta(B) = \sum_{\gamma \subset \alpha \otimes \beta} \Theta_\gamma((v_\gamma^{\alpha, \beta})^*(A \otimes B) v_\gamma^{\alpha, \beta}).$$

We can now give an explicit formula for the GNS representation π_h . Let x be a coefficient of u^α and let $\xi \in p_\beta L^2(\mathbb{G}) \simeq \mathcal{B}(H_\beta)$. Set $\hat{x} = \pi_h(x) \xi_h$, which is an element of $p_\alpha L^2(\mathbb{G}) \simeq \mathcal{B}(H_\alpha)$. Making the identification by Θ implicit, we have

$$(1) \quad \pi_h(x) \xi = \hat{x} \cdot \xi = \sum_{\gamma \subset \alpha \otimes \beta} (v_\gamma^{\alpha, \beta})^*(\tilde{x} \otimes \xi) v_\gamma^{\alpha, \beta} = \sum_{\gamma \subset \alpha \otimes \beta} \text{Ad}(v_\gamma^{\alpha, \beta})(\tilde{x} \otimes \xi).$$

2.4. Free quantum groups. We will be concerned in the sequel with the free unitary and free orthogonal quantum groups. They were first defined by A. Van Daele and S. Wang in [30, 33] and the definition was later slightly modified by T. Banica in [2]. This section is devoted to briefly recalling the definition and main properties of these free quantum groups. If A is a C^* -algebra and if $u = (u_{i,j})$ is a matrix with coefficients in A , we set $\bar{u} = (u_{i,j}^*)$.

Definition 2.7. Let $N \in \mathbb{N}$ and let $F \in GL_N(\mathbb{C})$ be such that $F \bar{F} \in \mathbb{R} \cdot \text{Id}$. We denote by $A_u(F)$ the universal unital C^* -algebra generated by N^2 elements $(u_{i,j})$ such that the matrices $u = (u_{i,j})$ and $F \bar{u} F^{-1}$ are *unitary*. Similarly, we denote by

$A_o(F)$ the universal unital C^* -algebra generated by N^2 elements $(v_{i,j})$ such that the matrix $v = (v_{i,j})$ is *unitary* and $v = F\bar{v}F^{-1}$.

One can easily check that there is a unique coproduct Δ_u (resp. Δ_o) on $A_u(F)$ (resp. $A_o(F)$) such that for all i, j ,

$$\begin{aligned}\Delta_u(u_{i,j}) &= \sum_{k=0}^N u_{i,k} \otimes u_{k,j} \\ \Delta_o(v_{i,j}) &= \sum_{k=0}^N v_{i,k} \otimes v_{k,j}\end{aligned}$$

Definition 2.8. A pair $(A_u(F), \Delta_u)$ is called a *free unitary quantum group* and will be denoted $U^+(F)$. A pair $(A_o(F), \Delta_o)$ is called a *free orthogonal quantum group* and will be denoted $O^+(F)$. Their discrete duals will be denoted respectively $\mathbb{F}U^+(F)$ and $\mathbb{F}O^+(F)$.

Remark 2.9. The restriction on the matrix F in the definition is equivalent to requiring the fundamental representation v of $O^+(F)$ to be irreducible. That assumption is necessary in order to get a nice description of the representation theory of $O^+(F)$.

Any *compact matrix pseudogroup* in the sense of [36, Def. 1.1] is a compact quantum subgroup of a free unitary quantum group. Moreover, if its fundamental corepresentation is equivalent to its contragredient, then it is a compact quantum subgroup of a free orthogonal quantum group. In this sense, we can see $U^+(F)$ and $O^+(F)$ as quantum generalizations of the usual unitary and orthogonal groups. The representation theory of free orthogonal quantum groups was computed by T. Banica in [2].

Theorem 2.10 (Banica). *The equivalence classes of irreducible representations of $O^+(F)$ are indexed by the set \mathbb{N} of integers (u^0 being the trivial representation and $u^1 = u$ the fundamental one), each one is isomorphic to its contragredient and the tensor product is given (inductively) by*

$$u^1 \otimes u^n = u^{n+1} \oplus u^{n-1}.$$

Moreover, if $N = 2$, then $\dim_q(u^n) = n + 1$. Otherwise,

$$\dim_q(u^n) = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}},$$

where $q + q^{-1} = \text{Tr}(Q_1)$ and $0 \leq q \leq 1$. We will use the shorthand notation D_n for $\dim_q(u^n)$ in the sequel.

Remark 2.11. The following inequality always holds : $q + q^{-1} \geq N$.

The representation theory of $U^+(F)$ was also explicitly computed by T. Banica in [3]. However, we will only need the following result [3, Thm. 1] (see [33] for the definition of the free product of discrete quantum groups).

Theorem 2.12 (Banica). *The discrete quantum group $\mathbb{F}U^+(F)$ is a quantum subgroup of $\mathbb{Z} * \mathbb{F}O^+(F)$.*

The following lemma summarizes some standard calculations which will be used several times in the sequel.

Lemma 2.13. *Let $a > b$ be integers, then $D_{a-b}^{-1} \leq D_b/D_a \leq q^{a-b}$. Moreover, for any integer c , $q^c D_c \leq (1 - q^2)^{-1}$.*

Proof. Let $n \in \mathbb{Z}$ such that $n \geq -b$. Decomposing $u^{b+n} \otimes u^{a+n+1}$ and $u^{b+n+1} \otimes u^{a+n}$ into sums of irreducible representations yields

$$D_{b+n} D_{a+n+1} = D_{a-b+1} + \cdots + D_{a+b+2n+1} \leq D_{a-b-1} + \cdots + D_{a+b+2n+1} = D_{b+n+1} D_{a+n}$$

This inequality means that the sequence $(D_{b+n}/D_{a+n})_{n \geq -b}$ is increasing, thus any term is greater than its first term D_{a-b}^{-1} and less than its limit q^{a-b} . The second part of the lemma is obvious since $q^c D_c = (1 - q^{2c+2})/(1 - q^2)$. \square

3. APPROXIMATION PROPERTIES FOR DISCRETE QUANTUM GROUPS

We now give some definitions and properties concerning weak amenability and the Haagerup property for discrete quantum groups. It is based on the notion of multipliers associated to bounded functions.

Definition 3.1. Let $\widehat{\mathbb{G}}$ be a discrete quantum group and $a \in \ell^\infty(\widehat{\mathbb{G}})$. The *left multiplier* associated to a is the map $m_a : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G})$ defined by

$$(m_a \otimes \iota)(u^\alpha) = (1 \otimes a p_\alpha) u^\alpha,$$

for any irreducible representation α of \mathbb{G} .

Remark 3.2. This definition relies on the identification of $\ell^\infty(\widehat{\mathbb{G}})$ with $\prod \mathcal{B}(H_\alpha)$ which is specific to the case of discrete quantum groups. However, since W reads as $\prod u^\alpha$ in this identification, we can equivalently define the multiplier m_a in the following way : for any $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$, $m_a((\iota \otimes \omega)(W)) = (\iota \otimes \omega)((1 \otimes a)W)$. This definition makes sense in a more general setting and corresponds to the definition of J. Kraus and Z.J. Ruan in [22] for Kac algebras and to the definition of M. Daws in [16] for locally compact quantum groups.

Definition 3.3. A net (a_i) of elements of $\ell^\infty(\widehat{\mathbb{G}})$ is said to *converge pointwise* to $a \in \ell^\infty(\widehat{\mathbb{G}})$ if $a_i p_\alpha \rightarrow a p_\alpha$ for any irreducible representation α of \mathbb{G} . An element $a \in \ell^\infty(\widehat{\mathbb{G}})$ is said to have *finite support* if $a p_\alpha$ is non-zero only for a finite number of irreducible representations α .

The key point to get a suitable definition of weak amenability is to have an intrinsic characterization of the completely bounded norm of a multiplier. Such a characterization is given by the following theorem [16, Prop 4.1 and Thm 4.2].

Theorem 3.4 (Daws). *Let $\widehat{\mathbb{G}}$ be a discrete quantum group and $a \in \ell^\infty(\widehat{\mathbb{G}})$. Then m_a extends to a completely bounded multiplier on $\mathcal{B}(L^2(\mathbb{G}))$ if and only if there exists a Hilbert space K and two maps $\xi, \eta \in \mathcal{B}(L^2(\mathbb{G}), L^2(\mathbb{G}) \otimes K)$ such that $\|\xi\|\|\eta\| = \|m_a\|_{cb}$ and*

$$(2) \quad (1 \otimes \eta)^* \widehat{W}_{12}^* (1 \otimes \xi) \widehat{W} = a \otimes 1.$$

Moreover, we then have $m_a(x) = \eta^*(x \otimes 1)\xi$.

Notice that thanks to this theorem, the completely bounded norm of m_a is the same when it is extended to $C_{\text{red}}(\mathbb{G})$, $L^\infty(\mathbb{G})$ or $\mathcal{B}(L^2(\mathbb{G}))$. Denoting by $\|m_a\|_{cb}$ this norm, we can give a definition of weak amenability.

Definition 3.5. A discrete quantum group $\widehat{\mathbb{G}}$ is said to be *weakly amenable* if there exists a net (a_i) of elements of $\ell^\infty(\widehat{\mathbb{G}})$ such that

- a_i has finite support for all i .
- (a_i) converges pointwise to 1.
- $K := \limsup_i \|m_{a_i}\|_{cb}$ is finite.

The lower bound of the constants K for all nets satisfying these properties is denoted $\Lambda_{cb}(\widehat{\mathbb{G}})$ and called the *Cowling-Haagerup constant* of $\widehat{\mathbb{G}}$. By convention, $\Lambda_{cb}(\widehat{\mathbb{G}}) = \infty$ if $\widehat{\mathbb{G}}$ is not weakly amenable.

It is clear on the definition that a discrete group G is weakly amenable in the classical sense (see e.g. [13, Def. 12.3.1]) if and only if the associated discrete quantum group is weakly amenable (and the constants are the same). We recall the following notions of weak amenability for operator algebras.

Definition 3.6. A C^* -algebra A is said to be *weakly amenable* if there exists a net (T_i) of linear maps from A to itself such that

- T_i has finite rank for all i .
- $\|T_i(x) - x\| \rightarrow 0$ for all $x \in A$.
- $K := \limsup_i \|T_i\|_{cb}$ is finite.

The lower bound of the constants K for all nets satisfying these properties is denoted $\Lambda_{cb}(A)$ and called the *Cowling-Haagerup constant* of A . By convention, $\Lambda_{cb}(A) = \infty$ if the C^* -algebra A is not weakly amenable.

A von Neumann algebra N is said to be *weakly amenable* if there exists a net (T_i) of normal linear maps from N to itself such that

- T_i has finite rank for all i .
- $(T_i(x) - x) \rightarrow 0$ ultraweakly for all $x \in N$.
- $K := \limsup_i \|T_i\|_{cb}$ is finite.

The lower bound of the constants K for all nets satisfying these properties is denoted $\Lambda_{cb}(N)$ and called the *Cowling-Haagerup constant* of N . By convention, $\Lambda_{cb}(N) = \infty$ if the von Neumann algebra N is not weakly amenable.

The following theorem is a generalization of a well-known result in the classical case. It was first proved for Kac algebras of discrete type by J. Kraus and Z.J. Ruan in [22, Thm. 5.14]. See [19, Thm. 3.11] for a proof in the general case.

Theorem 3.7. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group, then*

$$\Lambda_{cb}(\widehat{\mathbb{G}}) = \Lambda_{cb}(C_{red}(\mathbb{G})) = \Lambda_{cb}(L^\infty(\mathbb{G})).$$

The Haagerup property for discrete quantum groups is defined in the same way, mimicking the definition for discrete groups.

Definition 3.8. A discrete quantum group $\widehat{\mathbb{G}}$ is said to have the *Haagerup property* if there exists a net (a_i) of elements of $\ell^\infty(\widehat{\mathbb{G}})$ such that

- $a_i \in C_0(\widehat{\mathbb{G}})$ for all i .
- (a_i) converges pointwise to 1.
- m_{a_i} is unital and completely positive for all i .

Again there is a corresponding notion at the level of von Neumann algebras. Note however that there is up to now no good notion of the Haagerup property for general C^* -algebras.

Definition 3.9. A (non-necessarily finite) von Neumann algebra N is said to have the *Haagerup property with respect to a state σ* if there exists a net (T_i) of normal linear maps from N to itself such that

- $\sigma \circ T_i \leq \sigma$ and T_i extends to a compact map on $L^2(N, \sigma)$ for all i .
- $(T_i(x) - x) \rightarrow 0$ ultraweakly for all $x \in N$.
- T_i is completely positive for all i .

It follows from [15] that if m_a is completely positive, there is a state ω_a on $C_{\max}(\mathbb{G})$ such that $a = (\omega_a \otimes \iota)(W)$. This implies that m_a is completely positive. With this observation in mind, the proof of the following result is a slight modification of the proof of [19, Thm. 3.11].

Theorem 3.10. *A discrete quantum group $\widehat{\mathbb{G}}$ has the Haagerup property if and only if $L^\infty(\mathbb{G})$ has the Haagerup property relative to the Haar state h .*

We will also need in the end the following result, which is well-known at least in the finite case.

Proposition 3.11. *Let N_1 be a von Neumann algebra having the Haagerup property relative to a state σ_1 and let N_2 be another von Neumann algebra having the Haagerup property relative to a state σ_2 . Then, if M denotes the free product of N_1 and N_2 with respect to the states σ_1 and σ_2 , M has the Haagerup property relative to $\sigma_1 * \sigma_2$.*

Proof. Since the proof of [9, Thm 3.9] in the finite case nowhere makes use of the traciality of the states involved, it can be applied without any modification to this slightly more general setting. \square

4. THE PROJECTIONS ON IRREDUCIBLE REPRESENTATIONS IN FREE ORTHOGONAL QUANTUM GROUPS

This section contains our main technical result : we give a polynomial bound for the completely bounded norm of the projection on the linear span of coefficients of an irreducible representation u^d in $C_{\text{red}}(O^+(F))$. Let us give some motivation for this. First note that this projection is simply the multiplier m_{p_d} associated to $p_d \in \ell^\infty(\widehat{\mathbb{G}})$. If we choose for every integer k and real number t a scalar coefficient $b_k(t)$, we can define a net of (radial) elements

$$a_i(t) = \sum_{k=0}^i b_k(t) p_k \in \ell^\infty(\widehat{\mathbb{G}}).$$

If the $b_k(t)$ have sufficiently nice properties and if the completely bounded norm of the operators m_{p_d} can be controlled, the net $(a_i(t))$ will satisfy all the hypothesis in Definition 3.5 and $\mathbb{F}O^+(F)$ will be weakly amenable.

Our strategy to obtain the polynomial bound is inspired from the proof of U. Haagerup's estimate for the completely bounded norm of projections on words of fixed length in free groups. The original proof is unpublished but the argument is detailed in G. Pisier's book [27].

From now on, we fix an integer $N > 2$ and a matrix $F \in GL_N(\mathbb{C})$ satisfying $F\overline{F} \in \mathbb{R} \cdot \text{Id}$. We will write \mathcal{H} for the Hilbert space $L^2(O^+(F))$ which is identified to $\oplus_k \mathcal{B}(H_k)$ as explained in Subsection 2.3 (H_k being the carrier Hilbert space of the k -th irreducible representation). Let H be a fixed Hilbert space and let $X \in \mathcal{B}(H) \odot \text{Pol}(O^+(F))$ (it is enough to control the norm on this dense subalgebra), chose $d \in \mathbb{N}$ and set $X^d = (\iota \otimes m_{p_d})(X)$. These objects should be thought of as "operator-valued functions with finite support" on $\mathbb{F}O^+(F)$. Our aim is to control the norm of X^d using the norm of X .

Remark 4.1. Recall from [32] that there is a natural length on the discrete quantum group $\mathbb{F}O^+(F)$ such that the irreducible representation u_d has length d . Using this notion, one could give a rigorous definition of "operator-valued functions with support in the words of length d ". This, however, will not be needed here.

4.1. Block decomposition. We start by decomposing the operators into more elementary ones. For any two integers a and b , we set

$$\begin{aligned} B_{a,b}(X) &:= (\iota \otimes p_a)X(\iota \otimes p_b) \\ B_{a,b}(X^d) &:= (\iota \otimes p_a)X^d(\iota \otimes p_b) \end{aligned}$$

This is simply X (resp. X^d) seen as an operator from $\mathcal{B}(H_b)$ to $\mathcal{B}(H_a)$ and obviously has norm less than $\|X\|$ (resp. $\|X^d\|$). We will call it a *block*. The operator X^d admits a particular decomposition with respect to these blocks.

Lemma 4.2. *Set $X_j^d = \sum_{k=0}^{+\infty} B_{d-j+k, j+k}(X^d)$, then $X^d = \sum_{j=0}^d X_j^d$.*

Proof. Clearly, $X^d = \sum_{a,b} B_{a,b}(X^d)$. If we decompose X^d as $\sum_i T_i \otimes x_i$, with x_i a coefficient of u^d and $T_i \in \mathcal{B}(H)$, we see that X^d sends $H \otimes (p_b \mathcal{H})$ into $\oplus_c (H \otimes (p_c \mathcal{H}))$ where the sum runs over all irreducible subrepresentations c of $d \otimes b$. Thus, we deduce from Theorem 2.10 that $B_{a,b}(X^d)$ vanishes as soon as a is not of the form $d + b - 2j$ for some $0 \leq j \leq \min(d, b)$. Consequently,

$$X^d = \sum_{b=0}^{+\infty} \sum_{j=0}^{\min(d,b)} B_{d+b-2j,b}(X^d) = \sum_{j=0}^d \sum_{b=j}^{+\infty} B_{d+b-2j,b}(X^d).$$

Writing $b = k + j$, we get the desired result. □

This should be thought of as a decomposition according to the "number of deleted letters" in the action of X^d . Thanks to the triangle inequality, we can restrict ourselves to the study of $\|X_j^d\|$. Proposition 4.5 further reduces the problem to the study of only one specific block in X_j^d . Before stating and proving it, we have to introduce several notations and elementary facts.

Recall from Subsection 2.3 that for $\gamma \subset \alpha \otimes \beta$, $v_\gamma^{\alpha,\beta} : H_\gamma \mapsto H_\alpha \otimes H_\beta$ denotes an isometric intertwiner and let $M_k^+ : \mathcal{H} \otimes \mathcal{B}(H_k) \rightarrow \mathcal{H}$ be the orthogonal sum of the operators $\text{Ad}(v_{l+k}^{l,k})$. Under our identification of \mathcal{H} with $\oplus \mathcal{B}(H_k)$, the restriction of M_k^+ to $\mathcal{B}(H_l) \otimes \mathcal{B}(H_k)$ is just the map induced by the product composed with the orthogonal projection onto $\mathcal{B}(H_{l+k})$. If we endow $\mathcal{B}(H_k)$ with the scalar product $\langle \cdot, \cdot \rangle_k$, it can be seen as a subspace of \mathcal{H} and we can compute the norm of the restriction of M_k^+ to $\mathcal{B}(H_l) \otimes \mathcal{B}(H_k)$ with respect to the Hilbert structure on $\mathcal{H} \otimes \mathcal{H}$.

Let $x \in \mathcal{B}(H_l) \otimes \mathcal{B}(H_k)$, then

$$\begin{aligned}
 \|M_k^+(x)\|^2 &= \frac{1}{D_{l+k}} \text{Tr}(Q_{k+l} M_k^+(x)^* M_k^+(x)) \\
 &= \frac{1}{D_{l+k}} \text{Tr}(Q_{k+l} (v_{l+k}^{l,k})^* x^* v_{l+k}^{l,k} (v_{l+k}^{l,k})^* x v_{l+k}^{l,k}) \\
 &\leq \frac{1}{D_{l+k}} \text{Tr}(Q_{l+k} (v_{l+k}^{l,k})^* x^* x v_{l+k}^{l,k}) \\
 &= \frac{1}{D_{l+k}} \text{Tr}(v_{l+k}^{l,k} Q_{l+k} (v_{l+k}^{l,k})^* x^* x) \\
 &= \frac{1}{D_{l+k}} (\text{Tr} \otimes \text{Tr})((Q_l \otimes Q_k) x^* x) \\
 &= \frac{D_l D_k}{D_{l+k}} \|x\|^2
 \end{aligned}$$

i.e. $\|M_k^+(p_l \otimes \iota)\|^2 = D_l D_k / D_{l+k}$ (the norm is attained at $x = v_{l+k}^{l,k} (v_{l+k}^{l,k})^*$).

Remark 4.3. Note that this computation also proves that $\|M_k^+\|^2 = \frac{1 - q^{2k+2}}{1 - q^2}$ and in particular that $\|M_1^+\|^2 = 1 + q^2 \leq 2$.

Remark 4.4. The computation of the adjoint of M_k^+ is similar to the computation of the norm. One has $(M_k^+)^* p_{l+k} = (D_l D_k / D_{l+k}) \text{Ad}((v_{l+k}^{l,k})^*)$.

Let us now state and prove the main result of this subsection.

Proposition 4.5. *For integers a, b and c , set*

$$N_{a,b}^c = 1 - \frac{D_{(a-b+c)/2} D_{(b-a+c)/2-1}}{D_{a+1} D_b}$$

whenever this expression makes sense. Then if we set

$$\chi_j^d(k) = \sqrt{\frac{D_{d-j} D_{j+k}}{D_{d-j+k} D_j}} \prod_{i=0}^{j-1} (N_{d-j+i, k+i}^{d-j+k})^{-1},$$

we have, for all k , $\|B_{d-j+k, j+k}(X^d)\| \leq \chi_j^d(k) \|B_{d-j, j}(X^d)\|$.

Proof. Let us first focus on the one-dimensional case. Let x be a coefficient of u^d seen as an element of $\mathcal{B}(H_d)$ and choose an integer k . Let us compare the two operators

$$\begin{aligned}
 A &= [M_k^+(p_{d-j} x p_j \otimes \iota) (M_k^+)^*](\xi) \\
 B &= (p_{d-j+k} x p_{j+k})(\xi)
 \end{aligned}$$

for $\xi \in p_{j+k} \mathcal{H} = \mathcal{B}(H_{j+k})$. Setting $V = (\iota \otimes v_{j+k}^{j,k})^* (v_{d-j}^{d,j} \otimes \iota) v_{d-j+k}^{d-j,k}$, we have an intertwiner between u^{d-j+k} and $u^{d \otimes (j+k)}$. Since that inclusion has multiplicity one,

there is a complex number $\mu_j^d(k)$ such that

$$V = \mu_j^d(k) v_{d-j+k}^{d,j+k}.$$

Now, using Equation (1) and Remark 4.4, we have

$$\begin{aligned} B &= (v_{d-j+k}^{d,j+k})^* (x \otimes \xi) v_{d-j+k}^{d,j+k} \\ A &= V^* (x \otimes \xi) \left(\frac{D_j D_k}{D_{j+k}} \right) V \end{aligned}$$

and consequently $B = \lambda_j^d(k) A$, with $\lambda_j^d(k) = (D_j D_k / D_{j+k})^{-1} |\mu_j^d(k)|^{-2}$. Let us compute $|\mu_j^d(k)|$. If we set $v_+^{a,b} = (v_{a+b}^{a,b})^*$ and define two morphisms of representations

$$\begin{aligned} \mathcal{T}_A &= (v_+^{d-j,j} \otimes v_+^{j,0} \otimes \iota_k) (\iota_{d-j} \otimes t_j \otimes \iota_k) v_{d-j+k}^{d-j,k} \\ \mathcal{T}_B &= (v_+^{d-j,j} \otimes v_+^{j,k}) (\iota_{d-j} \otimes t_j \otimes \iota_k) v_{d-j+k}^{d-j,k} \end{aligned}$$

Up to some complex numbers of modulus 1,

$$\mathcal{T}_A = \|\mathcal{T}_A\| (v_{d-j}^{d,j} \otimes \iota) v_{d-j+k}^{d-j,k} \text{ and } \mathcal{T}_B = \|\mathcal{T}_B\| v_{d-j+k}^{d,j+k}.$$

Since moreover $(\iota \otimes v_{j+k}^{j,k})^* \mathcal{T}_A = \mathcal{T}_B$, we get $|\mu_j^d(k)|^2 = \|\mathcal{T}_B\|^2 / \|\mathcal{T}_A\|^2$. Thanks to [31, Prop. 2.3] and [32, Lem. 4.8], we can compute the norms of \mathcal{T}_A and \mathcal{T}_B and obtain

$$|\mu_j^d(k)|^2 = \prod_{i=0}^{j-1} \frac{N_{d-j+i,k+i}^{d-j+k}}{N_{d-j+i,i}^{d-j}} = \prod_{i=0}^{j-1} N_{d-j+i,k+i}^{d-j+k}.$$

Note that for $j = 0$, the above product is not defined. However, $\lambda_0^d(k) = 1$ since $\mathcal{T}_A = \mathcal{T}_B$ in that case. As $\lambda_j^d(k)$ does not depend on ξ , we have indeed proved the following equality in $\mathcal{B}(\mathcal{H})$:

$$p_{d-j+k} x p_{j+k} = \lambda_j^d(k) [M_k^+ (p_{d-j} x p_j \otimes \iota) (M_k^+)^*].$$

Now we go back to the operator-valued case. We have $X^d = \sum_i T_i \otimes x_i$, where $x_i \in \text{Pol}(O^+(F))$ is a coefficient of u^d and $T_i \in \mathcal{B}(H)$, hence

$$\lambda_j^d(k) [(\iota \otimes M_k^+) (B_{d-j,j}(X^d) \otimes \iota) (\iota \otimes M_k^+)^*] = B_{d-j+k,j+k}(X^d).$$

Using the norms of the restrictions of M_k^+ computed above, we get

$$\|B_{d-j+k,j+k}(X^d)\| \leq \lambda_j^d(k) \|(\iota \otimes M_k^+) B_{d-j,j}(X^d) (\iota \otimes M_k^+)^*\| \leq \chi_j^d(k) \|B_{d-j,j}(X^d)\|.$$

□

Corollary 4.6. *There is a constant $K(q)$, depending only on q , such that for any $d \in \mathbb{N}$ and $0 \leq j \leq d$, $\|X_j^d\| \leq K(q) \|B_{d-j,j}(X^d)\|$.*

Proof. According to Lemma 2.13, we have

$$\frac{D_{d-j}D_{k-1}}{D_{d-j+i+1}D_{k+i}} \leq q^{i+1}q^{i+1} = q^{2i+2},$$

thus $(N_{d-j+i,k+i}^{d-j+k})^{-1} \leq (1 - q^{2i+2})^{-1}$. Again by Lemma 2.13, $D_{d-j}/D_{d-j+k} \leq q^k$ and $D_{j+k}/D_j \leq D_k$, hence

$$\chi_j^d(k) \leq \sqrt{q^k D_k} \prod_{i=0}^{j-1} \frac{1}{1 - q^{2i+2}} \leq \frac{1}{\sqrt{1 - q^2}} \prod_{i=0}^{+\infty} \frac{1}{1 - q^{2i+2}} = K(q).$$

□

4.2. Completely bounded norm. We now want to find some polynomial P such that $\|X^d\| \leq P(d)\|X\|$. Thanks to Proposition 4.5, the problem reduces to finding a polynomial Q such that $\|B_{d-j,j}(X^d)\| \leq Q(d)\|X\|$. This will be done using the following recursion formula.

Proposition 4.7. *Set*

$$N_1^+ = \bigoplus_l \frac{D_{l+1}}{D_1 D_l} M_1^+(p_l \otimes \iota).$$

According to Remark 4.4, $(N_1^+)^*$ is the sum of the operators $\text{Ad}((v_{l+k}^{l,k})^*)$, . Then there are coefficients $C_j^d(s)$ such that for $0 \leq j \leq d$,

$$\begin{aligned} B_{d-j+1,j+1}(X) &= (\iota \otimes M_1^+)(B_{d-j,j}(X) \otimes \iota)(\iota \otimes N_1^+)^* \\ &= B_{d-j+1,j+1}(X^{d+2}) + \sum_{s=0}^{\min(j,d-j)} C_j^d(s) B_{d-j+1,j+1}(X^{d-2s}) \end{aligned}$$

Proof. The idea of the proof is similar to the one used in the proof of Proposition 4.5. We first consider the one-dimensional case. Let x be a coefficient of u^l seen as an element of $\mathcal{B}(H_l)$. Fix an element $\xi \in p_{j+1}\mathcal{H}$. Again, the operators

$$\begin{aligned} A &= [M_1^+(p_{d-j}xp_j \otimes \iota)(N_1^+)^*](\xi) \\ B &= (p_{d-j+1}x_l p_{j+1})(\xi) \end{aligned}$$

are proportional. Note that if $l > d + 2$, $l < |d - 2j|$ or $l - d$ is not even, both operators are 0. Note also that if $l = d + 2$, $A = 0$. The other values of l can be written $d - 2s$ for some positive integer s between 0 and $\min(j, d - j)$. In that case, the existence of a scalar $\nu_j^d(s)$ such that $B = \nu_j^d(s)A$ follows from the same argument as in the proof of Proposition 4.5. Let us compute $\nu_j^d(s)$, noticing that thanks to the normalization of N_1^+ , the constant $\nu_j^d(s)$ only corresponds to the " μ -part" of the constant λ of Proposition 4.5. This time we have to set

$$\begin{aligned} \mathcal{T}_A &= (v_+^{d-s-j,j-s} \otimes v_+^{j-s,s} \otimes \iota_1)(\iota_{d-j-s} \otimes t_{j-s} \otimes \iota_{s+1})v_{d-j+1}^{d-j-s,s+1} \\ \mathcal{T}_B &= (v_+^{d-s-j,j-s} \otimes v_+^{j-s,s+1})(\iota_{d-j-s} \otimes t_{j-s} \otimes \iota_{s+1})v_{d-j+1}^{d-j-s,s+1} \end{aligned}$$

Again, applying [31, Prop. 2.3] and [32, Lem. 4.8] yields

$$\nu_j^d(s) = \frac{\|\mathcal{T}_A\|^2}{\|\mathcal{T}_B\|^2} = \prod_{i=0}^{j-s-1} \frac{N_{d-s-j+i,s+i}^{d-j}}{N_{d-j-s+i,s+i+1}^{d-j+1}}.$$

Like in the proof of Proposition 4.5, we can now go back to the operator-valued case. We have

$$X = \sum_l \sum_{i=0}^{k(l)} T_l^{(i)} \otimes x_l^{(i)}$$

where $x_l^{(i)} \in \text{Pol}(O^+(F))$ are coefficients of u^l and $T_l^{(i)} \in \mathcal{B}(H)$. Setting

$$X^l = \sum_{i=0}^{k(l)} T_l^{(i)} \otimes x_l^{(i)},$$

we have

$$B_{d-j+1,j+1}(X^l) = \nu_j^d(s)(\iota \otimes M_1^+)(B_{d-j,j}(X^l) \otimes \iota)(\iota \otimes N_1^+)^*$$

and setting $C_j^d(s) = 1 - \nu_j^d(s)^{-1}$ concludes the proof. \square

The last result we need is a control on the coefficients $C_j^d(s)$ and $\chi_j^d(s)$.

Lemma 4.8. *For any $0 \leq j \leq d$, $\sum_{s=0}^{\min(j,d-j)} |C_j^d(s)| \chi_{j-s}^{d-2s}(s+1) \leq 1$.*

Proof. We first give another expression of $|C_j^d(s)|$. Decomposing into sums of irreducible representations yields

$$D_{d-s-j+i+1}D_{s+i+1} - D_{d-s-j}D_s = D_{d-j+2} + \cdots + D_{d-j+2i+2} = D_i D_{d-j+i+2}$$

$$D_{d-s-j+i+1}D_{s+i} - D_{d-s-j}D_{s-1} = D_{d-j+1} + \cdots + D_{d-j+2i+1} = D_i D_{d-j+i+1}$$

which implies that

$$N_{d-j-s+i,s+i+1}^{d-j+1} = \frac{D_i D_{d-j+i+2}}{D_{d-s-j+i+1} D_{s+i+1}} \text{ and } N_{d-s-j+i,s+i}^{d-j} = \frac{D_i D_{d-j+i+1}}{D_{d-s-j+i+1} D_{s+i}}.$$

Hence

$$\nu_j^d(s) = \prod_{i=0}^{j-s-1} \frac{N_{d-s-j+i,s+i}^{d-j}}{N_{d-j-s+i,s+i+1}^{d-j+1}} = \prod_{i=0}^{j-s-1} \frac{D_{d-j+i+1} D_{s+i+1}}{D_{s+i} D_{d-j+i+2}} = \frac{D_j D_{d-j+1}}{D_s D_{d-s+1}}.$$

Again, noticing that $D_j D_{d-j+1} - D_s D_{d-s+1} = D_{d-j-s} D_{j-s-1}$ yields

$$|C_j^d(s)| = |1 - \nu_j^d(s)^{-1}| = \frac{D_{d-j-s} D_{j-s-1}}{D_{d-j+1} D_j}.$$

According to Lemma 2.13, we thus have

$$|C_j^d(s)| \leq q^{s+1} q^{s+1} = q^{2s+2}$$

Now we turn to $\chi_{j-s}^{d-2s}(s+1)$. In fact, we are going to bound $\chi_j^d(s+1)$ independantly of d and j . Decomposing into sums of irreducible representations, we get

$$D_{d-j+i+1}D_{k+i} - D_{d-j}D_{k-1} = D_{d-j+k+1} + \cdots + D_{d-j+k+2i+1} = D_i D_{d-j+k+i+1},$$

which implies that $N_{d-j+i,k+i}^{d-j+k} = D_i D_{d-j+k+i+1} / D_{d-j+i+1} D_{k+i}$. Now we can compute

$$\begin{aligned} \frac{\chi_j^d(s+1)}{\chi_j^d(s)} &= \sqrt{\frac{D_{j+s+1}D_{d-j+s}}{D_{j+s}D_{d-j+s+1}}} \prod_{i=0}^{j-1} \frac{D_{s+1+i}D_{d-j+s+i+1}}{D_{s+i}D_{d-j+s+i+2}} \\ &= \sqrt{\frac{D_{j+s+1}D_{d-j+s}}{D_{j+s}D_{d-j+s+1}}} \frac{D_{j+s}D_{d-j+s+1}}{D_s D_{d+s+1}} \\ &= \frac{\sqrt{D_{j+s}D_{d-j+s+1}D_{d-j+s}D_{j+s+1}}}{D_s D_{d+s+1}}. \end{aligned}$$

Using Lemma 2.13 again, we get

$$\frac{\chi_j^d(s+1)}{\chi_j^d(s)} \leq \sqrt{q^j D_j q^{d-j} D_{d-j}} \leq \frac{1}{1-q^2}.$$

Since $\chi_j^d(1) \leq (1-q^2)^{-1}$, we have proved that $\chi_j^d(s+1) \leq (1-q^2)^{-s-1}$. This bound is independant of d and j , thus it also works for $\chi_{j-s}^{d-2s}(s+1)$. Combining this with our previous estimate we can compute

$$\sum_{s=0}^{\min(j,d-j)} |C_j^d(s)| \chi_{j-s}^{d-2s}(s+1) \leq \sum_{s=0}^{+\infty} \left(\frac{q^2}{1-q^2} \right)^{s+1} = \frac{q^2}{1-2q^2}.$$

The last term is less than 1 as soon as $q \leq 1/\sqrt{3}$, hence in particular for any q such that $q + q^{-1} \geq 3$. \square

Gathering all our results will now give the estimate we need. To make things more clear, we will proceed in two steps. First we bound the norms of the blocks of X^d .

Proposition 4.9. *There exists a polynomial Q such that for any integer d and $0 \leq j \leq d$, $\|B_{d-j,j}(X^d)\| \leq Q(d)\|X\|$.*

Proof. First note that $B_{d,0}(X^d) = B_{d,0}(X)$ and $B_{0,d}(X^d) = B_{0,d}(X)$, hence we only have to consider the case $1 \leq j \leq d-1$. Moreover, applying the triangle inequality to the recursion relation of Proposition 4.7 yields

$$\begin{aligned} \|B_{d-j+1,j+1}(X^{d+2})\| &\leq (1 + \|M_1^+\| \|N_1^+\|) \|X\| \\ &\quad + \sum_{s=0}^{\min(j,d-j)} |C_j^d(s)| \|B_{d-j+1,j+1}(X^{d-2s})\|. \end{aligned}$$

We proceed by induction, with the following induction hypothesis $H(d)$: "For any integer $l \leq d$ and any $0 \leq j \leq l$, $\|B_{l-j,j}(X^l)\| \leq Q(l)\|X\|$ with $Q(X) = 2X + 1$ ". Because of the remark at the beginning of the proof, $H(0)$ and $H(1)$ are true. Knowing this, we just have to prove that for any d , $H(d)$ implies the inequality for $d + 2$. Indeed, this will prove that assuming $H(d)$, both the inequalities for $d + 1$ (noticing that $H(d)$ implies $H(d - 1)$) and $d + 2$ are true, hence $H(d + 2)$ will hold.

Assume $H(d)$ to be true for some d and apply the recursion formula above. The blocks in the right-hand side of the inequality are of the form $B_{d-j+1,j+1}(X^{d-2s})$. By Proposition 4.5 and $H(d)$,

$$\begin{aligned} \|B_{d-j+1,j+1}(X^{d-2s})\| &= \|B_{(d-2s)-(j-s)+s+1,(j-s)+s+1}(X^{d-2s})\| \\ &\leq \chi_{j-s}^{d-2s}(s+1) \|B_{(d-2s)-(j-s),(j-s)}(X^{d-2s})\| \\ &\leq \chi_{j-s}^{d-2s}(s+1) Q(d-2s) \|X\|. \end{aligned}$$

Then, bounding $Q(d - 2s)$ by $Q(d)$ and using Lemma 4.8 yields

$$\|B_{d-j+1,j+1}(X^{d+2})\| \leq 3\|X\| + Q(d)\|X\| \leq Q(d+2)\|X\|.$$

Since $\|B_{d-j+1,j+1}(X^{d+2})\| = \|B_{(d+2)-(j+1),j+1}(X^{d+2})\|$, the inequality is proved for $1 \leq j + 1 \leq d + 1$. In other words, we have $\|B_{d-J,J}(X^{d+2})\| \leq Q(d+2)\|X\|$ for any $1 \leq J \leq d + 1$. As noted at the beginning of the proof, this is enough to get $H(d + 2)$. \square

Secondly we bound the norm of X^d itself.

Theorem 4.10. *There exists a polynomial P such that for all integers d ,*

$$\|m_{p_d}\|_{cb} \leq P(d).$$

Proof. We use the notations of Proposition 4.9. We know from Corollary 4.6 that $\|X_j^d\| \leq K(q)\|B_{d-j,j}(X^d)\|$, thus $\|X_j^d\| \leq K(q)Q(d)\|X\|$. If we set $P(X) = K(q)(X + 1)Q(X)$, we get $\|X^d\| \leq P(d)\|X\|$ by applying the triangle inequality to the decomposition of Lemma 4.2. \square

It is known that in the free group case, the completely bounded norm of the projections on words of fixed length grows exactly linearly (see e.g. [21]). Our technique cannot determine whether such a result still holds in the quantum case but proves the slightly weaker fact that the growth is at most quadratic. However, we can prove that it is at least linear. Let us first recall that the sequence (μ_k) of (dilated) Chebyshev polynomials of the second kind is defined by $\mu_0(X) = 1$, $\mu_1(X) = X$ and

$$X\mu_k(X) = \mu_{k-1}(X) + \mu_{k+1}(X)$$

Proposition 4.11. *There exists a polynomial R of degree one such that*

$$\|m_{p_d}\|_{cb} \geq R(d).$$

Proof. Since $\|m_{p_d}\|_{cb} \geq \|m_{p_d}\|$, we will simply prove a lower bound for this second norm. Let $\chi_n \in \text{Pol}(\mathbb{G})$ be the character of the representation u^n , i.e.

$$\chi_n = (\iota \otimes \text{Tr})(u_n).$$

Our aim is to prove that looking at the action of m_{p_d} on $\chi_{d+2} - \chi_d$ is enough to get the lower bound.

It is known (see [2]) that sending χ_n to the restriction to $[-2, 2]$ of the n -th Tchebyshev polynomial of the second kind μ_n yields an isomorphism between the sub-C*-algebra of $C_{\text{red}}(O^+(F))$ generated by the elements χ_n and $C([-2, 2])$. Moreover, the restriction of these polynomials to the interval $[-2, 2]$ form a Hilbert basis with respect to the scalar product associated to the semicircular law

$$d\nu = \frac{\sqrt{4-t^2}}{2\pi} dt.$$

Let us denote by $\pi : C([-2, 2]) \rightarrow \mathcal{B}(L^2([-2, 2], d\nu))$ the faithful representation by multiplication operators. What precedes means precisely that we have, for any finite sequence (a_n) ,

$$\left\| \sum a_n \chi_n \right\|_{C_{\text{red}}(O_N^+)} = \left\| \sum a_n \pi(\mu_n) \right\|_{\mathcal{B}(L^2([-2, 2], d\nu))}.$$

Let e_i denote the image of μ_i in $L^2([-2, 2], d\nu)$ and denote by T_n the operator sending e_i to e_{i+n} for $n \geq 0$. Letting E_j denote the linear span of the vectors e_i for $0 \leq i \leq j$, we can also define operators T_{-n} which are 0 on E_{n-1} and send e_i to e_{i-n} for $i \geq n$. The last operator we need, denoted S_n , sends $e_i \in E_n$ to e_{n-i} and is 0 on E_n^\perp . These translation operators obviously have norm 1. Moreover, a simple computation using Theorem 2.10 (or equivalently the recursion relation of the Chebyshev polynomials) shows that

$$\pi(\mu_{n+2} - \mu_n) = T_{n+2} - S_n - T_{-(n+2)}.$$

Thus $\|\chi_{n+2} - \chi_n\| = \|\pi(\mu_{n+2} - \mu_n)\| \leq 3$. On the other hand, it easily seen that $\mu_n(2) = n + 1$. In fact, this is true for $\mu_1(X) = X$ and $\mu_2(X) = X^2 - 1$ and we have the recursion relation

$$2\mu_n(2) = \mu_{n+1}(2) + \mu_{n-1}(2).$$

This implies that $\|\chi_n\| = \|\mu_n\|_\infty \geq n + 1$. Combining these two facts yields

$$\|m_{p_d}\| \geq \frac{\|m_{p_d}(\chi_{d+2} - \chi_d)\|}{\|\chi_{d+2} - \chi_d\|} \geq \frac{d+1}{3}$$

and setting $R(X) = (X + 1)/3$ concludes the proof. \square

5. THE MAIN THEOREM

In this section we will be concerned with approximation properties for free quantum groups. Our aim is to prove the Haagerup approximation property in the non-unimodular case and then use it in combination with the results of the previous section to obtain weak amenability. When $F = I_N$ M. Brannan gave in [10] a proof of the Haagerup property relying on the existence of a bounded antipode (which can be easily extended to the case where F is unitary, see below). We are going to adapt his argument to the general case where the antipode is unbounded.

To do so, we need a few additional preliminaries about representation theory. For every irreducible representation $\alpha \in \text{Irr}(\mathbb{G})$, we fix an orthonormal basis e_i^α of H_α which diagonalizes the matrix $Q_\alpha = j_\alpha^* j_\alpha$, i.e.

$$Q_\alpha e_i^\alpha = \lambda_i^\alpha e_i^\alpha.$$

This implies that $\|j_\alpha(e_i^\alpha)\|^2 = \lambda_i^\alpha$ and thus, setting $e_i^{\bar{\alpha}} = (\lambda_i^\alpha)^{-1/2} j_\alpha(e_i^\alpha)$ coherently defines an orthonormal basis of $H_{\bar{\alpha}}$ diagonalizing the matrix $Q_{\bar{\alpha}} = j_{\bar{\alpha}}^* j_{\bar{\alpha}} = (j_\alpha^{-1})^* j_\alpha^{-1}$. In fact, we have

$$Q_{\bar{\alpha}} e_i^{\bar{\alpha}} = (\lambda_i^\alpha)^{-1} e_i^{\bar{\alpha}}.$$

Let $u_{i,j}^\alpha$ be the (i, j) -th coefficient of u^α in the previous basis. Then, we have

$$u_{i,j}^{\bar{\alpha}} = \sqrt{\frac{\lambda_j^\alpha}{\lambda_i^\alpha}} (u_{i,j}^\alpha)^*$$

and, in particular, $(u_{k,k}^\alpha)^* = u_{k,k}^{\bar{\alpha}}$.

5.1. Twisted characters and the averaging principle. We now extend the averaging principle used by M. Brannan [10, Thm 3.7] to the non-unimodular setting. In this section, we still work in full generality with an arbitrary compact quantum group \mathbb{G} . If φ is a state on say $C(\mathbb{G})$, one can define the *(left) convolution operator* C_φ associated to φ by

$$C_\varphi = (\varphi \otimes \iota) \circ \Delta.$$

This operator makes sense as a u.c.p. map on $C(\mathbb{G})$ and extends to a normal map on $L^\infty(\mathbb{G})$ (see e.g. [10, Lem 3.4]). Moreover, by invariance of the Haar state, it also yields a u.c.p. map on the reduced C^* -algebra $C_{\text{red}}(\mathbb{G})$. The basic idea in [10], which had already been used in [22], is to average a convolution operator using the antipode in such a way that the resulting operator only depends on the restriction of the initial state φ to some "easy-to-handle" C^* -subalgebra. As we will see, this algebra is not in general the algebra of characters used by M. Brannan but an algebra of *twisted characters*. The unboundedness of the antipode in the general case seems an obstruction to this strategy. However, it is quite natural to try the unitary antipode R as a substitute and our choice of the basis e_i^α is intended to

make the computation of R easier on coefficients of irreducible representations. In fact, let τ_z be the scaling group of $O^+(F)$, which is defined by

$$(\tau_z \otimes \iota)(u^\alpha) = (1 \otimes Q_\alpha^{iz})u^\alpha(1 \otimes Q_\alpha^{-iz}).$$

In our chosen basis, $\tau_{i/2}(u_{a,b}^\alpha) = (\lambda_b^\alpha)^{1/2}(\lambda_a^\alpha)^{-1/2}u_{a,b}^\alpha$ and consequently,

$$(3) \quad R(u_{a,b}^\alpha) = S \circ \tau_{i/2}(u_{a,b}^\alpha) = \sqrt{\frac{\lambda_b^\alpha}{\lambda_a^\alpha}}(u_{b,a}^\alpha)^*.$$

Let φ be a state on $C(\mathbb{G})$. Note the the coproduct Δ extends to an isometry from $L^2(\mathbb{G})$ to $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ and that its adjoint operator Δ^* maps $\text{Pol}(\mathbb{G}) \otimes \text{Pol}(\mathbb{G})$ into $\text{Pol}(\mathbb{G})$. Thus, the following "averaged" map

$$T_\varphi = \Delta^* \circ [(R \circ C_\varphi \circ R) \otimes \iota] \circ \Delta$$

is well defined on $C_{\text{red}}(\mathbb{G})$ and extends to a normal map on $L^\infty(\mathbb{G})$.

Proposition 5.1. *With the previous notations, the map T_φ is a u.c.p. h -preserving map and satisfies*

$$T_\varphi(u_{i,j}^\alpha) = \frac{1}{\dim_q(u^\alpha)} \left(\sum_k \varphi(\lambda_k^{\bar{\alpha}} u_{k,k}^{\bar{\alpha}}) \right) u_{i,j}^\alpha$$

Proof. The maps C_φ , R , Δ and Δ^* are obviously h -preserving so that T_φ is h -preserving. Moreover, C_φ and R are u.c.p and Δ is an isometry, thus T_φ is u.c.p.

Using Equation (3), we see that

$$(R \circ C_\varphi \circ R)(u_{i,k}^\alpha) = \sum_l \varphi((u_{k,l}^\alpha)^*) \sqrt{\frac{\lambda_k^\alpha}{\lambda_l^\alpha}} u_{i,l}^\alpha,$$

which yields

$$[(R \circ C_\varphi \circ R) \otimes \iota] \circ \Delta(u_{i,j}^\alpha) = \sum_{k,l} \varphi((u_{k,l}^\alpha)^*) \sqrt{\frac{\lambda_k^\alpha}{\lambda_l^\alpha}} (u_{i,l}^\alpha \otimes u_{k,j}^\alpha).$$

In the basis we chose, Schur's orthogonality relations read

$$\langle u_{i,j}^\alpha, u_{a,b}^\beta \rangle = h((u_{i,j}^\alpha)^* u_{a,b}^\beta) = \frac{\delta_{\alpha,\beta} \delta_{j,b}}{\dim_q(u^\alpha)} (Q_\alpha^{-1})_{i,a} = \frac{\delta_{\alpha,\beta} \delta_{i,a} \delta_{j,b}}{\dim_q(u^\alpha)} (\lambda_i^\alpha)^{-1},$$

hence we have

$$\begin{aligned} \langle \Delta^* \circ [(R \circ C_\varphi \circ R) \otimes \iota] \circ \Delta(u_{i,j}^\alpha), u_{a,b}^\beta \rangle &= \frac{\delta_{\alpha,\beta} \delta_{i,a} \delta_{j,b}}{\lambda_i^\alpha \dim_q(u^\alpha)^2} \sum_k (\lambda_k^\alpha)^{-1} \varphi((u_{k,k}^\alpha)^*) \\ &= \frac{\delta_{\alpha,\beta} \delta_{i,a} \delta_{j,b}}{\lambda_i^\alpha \dim_q(u^\alpha)^2} \sum_k \varphi(\lambda_k^{\bar{\alpha}} u_{k,k}^{\bar{\alpha}}). \end{aligned}$$

The coefficients of the irreducible representations forming an orthogonal system, we finally get

$$\begin{aligned} T_\varphi(u_{i,j}^\alpha) &= \frac{1}{\|u_{i,j}^\alpha\|^2} \left(\frac{\delta_{\alpha,\beta} \delta_{i,a} \delta_{j,b}}{\lambda_i^\alpha \dim_q(u^\alpha)^2} \sum_k \varphi(\lambda_k^\alpha u_{k,k}^\alpha) \right) u_{i,j}^\alpha \\ &= \frac{1}{\dim_q(u^\alpha)} \left(\sum_k \varphi(\lambda_k^\alpha u_{k,k}^\alpha) \right) u_{i,j}^\alpha \end{aligned}$$

□

Define the *twisted character* of an irreducible representation $\alpha \in \text{Irr}(\mathbb{G})$ to be the element

$$\tilde{\chi}_\alpha = (\text{Tr} \otimes \iota)((1 \otimes Q_\alpha)u^\alpha) = \sum_k \lambda_k^\alpha u_{k,k}^\alpha \in \text{Pol}(\mathbb{G}).$$

From the fact that Q_α commutes to morphisms, we see that the twisted characters only depend on the equivalence class of the irreducible representations u^α . The result of Proposition 5.1 can be rewritten

$$(4) \quad T_\varphi(u_{i,j}^\alpha) = \frac{\varphi(\tilde{\chi}_\alpha)}{\dim_q(u^\alpha)} u_{i,j}^\alpha.$$

This means that the operator T_φ only depends on the value of the state φ on the twisted characters. As soon as the quantum group is not unimodular, we have a whole bunch of states on the C*-algebra $C_{\max}(\mathbb{G})$ which can be easily computed on the twisted characters, namely the Woronowicz characters on the imaginary line. Recall that for any $z \in \mathbb{C}$, we can define a multiplicative linear map f_z on $\text{Pol}(\mathbb{G})$ by

$$(f_z \otimes \iota)(u^\alpha) = Q_\alpha^z.$$

It follows from [37, Thm 1.5] that if $z = it$ for some $t \in \mathbb{R}$, f_z extends to a *-character (and in particular a state) on the C*-algebra $C_{\max}(\mathbb{G})$. Moreover, a simple calculation yields

$$f_z(\tilde{\chi}_\alpha) = \text{Tr}(Q_\alpha^{1+it}).$$

5.2. Application to free quantum groups. The last step towards the proof of the Haagerup property is to investigate the algebraic relations between the twisted characters. In the case of free quantum groups, these algebraic relations are easily described, again thanks to the Chebyshev polynomials.

Proposition 5.2. *The twisted characters of irreducible representations satisfy*

$$\tilde{\chi}_1 \tilde{\chi}_k = \tilde{\chi}_{k-1} + \tilde{\chi}_{k+1}.$$

In other words, $\tilde{\chi}_k = \mu_k(\tilde{\chi}_1)$.

Proof. Recall that $v_a^{b,c}$ denotes an isometric intertwiner between the representations a and $b \otimes c$ and that for any morphism $T \in \text{Mor}(a, b \otimes c)$, $(Q_b \otimes Q_c)T = TQ_a$. Using these facts, we compute

$$\begin{aligned}
 \tilde{\chi}_1 \tilde{\chi}_k &= (\text{Tr} \otimes \text{Tr} \otimes \iota)((1 \otimes Q_1 \otimes Q_k)u_{13}^1 u_{23}^k) \\
 &= (\text{Tr} \otimes \text{Tr} \otimes \iota)((1 \otimes Q_1 \otimes Q_k)(1 \otimes v_{k-1}^{1,k})u^{k-1}(1 \otimes v_{k-1}^{1,k})^*) \\
 &+ (\text{Tr} \otimes \text{Tr} \otimes \iota)((1 \otimes Q_1 \otimes Q_k)(1 \otimes v_{k+1}^{1,k})u^{k+1}(1 \otimes v_{k+1}^{1,k})^*) \\
 &= (\text{Tr} \otimes \iota)([(1 \otimes v_{k-1}^{1,k})^*(1 \otimes Q_1 \otimes Q_k)(1 \otimes v_{k-1}^{1,k})]u^{k-1}) \\
 &+ (\text{Tr} \otimes \iota)([(1 \otimes v_{k+1}^{1,k})^*(1 \otimes Q_1 \otimes Q_k)(1 \otimes v_{k+1}^{1,k})]u^{k+1}) \\
 &= (\text{Tr} \otimes \iota)((1 \otimes Q_{k-1})u^{k-1}) + (\text{Tr} \otimes \iota)((1 \otimes Q_{k+1})u^{k+1}) \\
 &= \tilde{\chi}_{k-1} + \tilde{\chi}_{k+1}
 \end{aligned}$$

□

In order to simplify notations, we will use the fact that $n \simeq \bar{n}$ to replace $\tilde{\chi}_{\bar{n}}$ by $\tilde{\chi}_n$ in Equation 4. For $t \in \mathbb{R}$, set $b_k(t) = \mu_k(f_{it}(\tilde{\chi}_1))/\mu_k(D_1)$ and define elements

$$a(t) = \sum_{k=0}^{\infty} b_k(t)p_k \in \ell^\infty(\hat{\mathbb{G}}).$$

Proposition 5.2 precisely means that $T_{f_{it}} = m_{a_t}$.

Before going any further, we will settle the unimodular case since we cannot use the Woronowicz characters in that setting. Recall that $\mathbb{F}O^+(F)$ is unimodular if and only if F is a scalar multiple of a unitary matrix.

Proposition 5.3. *Let $F \in GL_N(\mathbb{C})$ be unitary up to a scalar. Then, the discrete quantum group $\mathbb{F}O^+(F)$ has the Haagerup property.*

Proof. According to [35] (see also [8, Rmk 5.7]), there is, up to isomorphism of the associated quantum groups, only two matrices F to consider, namely Id_N and (when N is even)

$$F = J_N = \begin{pmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{pmatrix}.$$

We claim that in both cases, setting $b_k(t) = \mu_k(t)/\mu_k(N)$ yields the Haagerup property. For $F = \text{Id}_N$ this is precisely [10, Thm 4.5]. For $F = J_N$, the arguments of [10] also apply and the only thing we have to prove is that N is not isolated in the spectrum of χ_1 in the C^* -algebra $C_{\max}(O^+(F))$. In fact, if $\theta \in \mathbb{R}$, the matrix

$$U^\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & \text{Id}_{N-2} \end{pmatrix}$$

is orthogonal and commutes to F (note that we can assume $N \geq 4$). Thus, by universality, $u_{i,j} \mapsto U_{i,j}^\theta$ extends to a well defined character on $A_o(J_N)$ sending χ_1

to $\text{Tr}(U^\theta) = N - 2 + 2\cos(\theta)$. This proves that $[N - 2, N]$ is contained in the spectrum of χ_1 . \square

We are now ready to extend the proof of the Haagerup property to the non-unimodular case and apply it to weak amenability. Note that replacing N by D_1 in the proof of [10, Prop 4.4] and using the fact that $|\mu_k(z)| \leq \mu_k(|z|)$ we get the following

Lemma 5.4. *Fix $2 < t_0 < 3$. Then, there exists a constant K_0 depending only on t_0 such that for any $z \in \mathbb{C}$ satisfying $t_0 < |z| < D_1$,*

$$\left| \frac{\mu_k(z)}{\mu_k(D_1)} \right| \leq K_0 \left(\frac{|z|}{D_1} \right)^k.$$

Theorem 5.5. *Let $N \geq 2$ and let $F \in GL_N(\mathbb{C})$ satisfy $F\overline{F} \in \mathbb{R} \cdot \text{Id}$. Then, the discrete quantum groups $\mathbb{F}O^+(F)$ and $\mathbb{F}U^+(F)$ have the Haagerup property. Moreover, the discrete quantum groups $\mathbb{F}O^+(F)$ and $\mathbb{F}U^+(F)$ are weakly amenable with Cowling-Haagerup constant equal to 1.*

Proof. Let us first consider the orthogonal case. For $N = 2$, the results are already known by amenability of the discrete quantum groups $\mathbb{F}O^+(F) = \widehat{SU_q(2)}$. Thus, we can assume $N > 2$. It is clear that $a(t)$ converges pointwise to 1 as t tends to 0. Observe that when t tends to 0, $g(t) = |\text{Tr}(Q_1^{1+it})|$ tends to $D_1 \geq 3$. Hence, there is some $t_1 > 0$ such that for $0 < t < t_1$, $t_0 \leq g(t) \leq D_1$. Moreover, $g(t)$ is strictly less than D_1 as soon as Q_1 has at least one eigenvalue which is not equal to 1, i.e. when F is not unitary. We can thus deduce from Lemma 5.4 that the elements $a(t)$ belong to $C_0(\widehat{\mathbb{G}})$ for $0 < t < t_1$. Since $m_{a(t)} = T_{f_{it}}$ which is u.c.p. by construction, we get the Haagerup property for $O^+(F)$. According to Proposition 3.11, $\mathbb{Z} * \mathbb{F}O^+(F)$ also has the Haagerup property and by Theorem 2.12, $\mathbb{F}U^+(F)$ has the Haagerup property (this is of course also true in the unimodular case).

Let us now consider the following truncated elements for any $\mathbb{F}O^+(F)$ (regardless of unimodularity)

$$a_i(t) = \sum_{k=0}^i b_k(t) p_k \in \ell^\infty(\mathbb{F}O^+(F)).$$

They form a net of finite rank elements converging pointwise to the identity and we just have to prove that the completely bounded norms of the associated multipliers satisfy the boundedness condition. In the non unimodular case, we have by Lemma 5.4 that for any $0 < t < t_1$,

$$\|m_{a(t)} - m_{a_i(t)}\|_{cb} \leq \sum_{k>i} K_0 \left(\frac{g(t)}{D_1} \right)^k \|m_{p_k}\|_{cb}.$$

In the unimodular case, the same inequality holds replacing $g(t)$ by t . The sum on the right-hand side tends to 0 as i goes to infinity since Theorem 4.10 implies that it

is the rest of an absolutely converging series. This implies that $\limsup \|m_{a_i(t)}\|_{cb} = 1$. In other words, $\Lambda_{cb}(\mathbb{F}O^+(F)) = 1$. By [20, Thm. 4.2], we also have $\Lambda_{cb}(\mathbb{Z} * \mathbb{F}O^+(F)) = 1$, hence $\Lambda_{cb}(\mathbb{F}U^+(F)) = 1$ by Theorem 2.12. \square

Remark 5.6. It is a consequence of Theorem 5.5 and [12, Thm 3.9] that $\mathbb{F}O^+(F)$ and $\mathbb{F}U^+(F)$ are exact because they are weakly amenable. This had previously been proved by S. Vaes and R. Vergnioux in [29] using an argument of amenability of the boundary action.

For the sake of completeness, let us explain how this weak amenability result extends to free quantum groups associated to arbitrary matrices F (the definition being exactly the same as Definition 2.7).

Corollary 5.7. *Let $F \in GL_N(\mathbb{C})$ be any matrix, then the discrete quantum groups $\mathbb{F}O^+(F)$ and $\mathbb{F}U^+(F)$ have the Haagerup property. Moreover, they are weakly amenable and have Cowling-Haagerup constant equal to 1.*

Proof. S. Wang proved in [35] that for a general F , $\mathbb{F}U^+(F)$ and $\mathbb{F}O^+(F)$ can be decomposed as free product (without amalgamation) of free orthogonal and free unitary quantum groups for matrices F' satisfying $F'\overline{F'} \in \mathbb{R} \cdot \text{Id}$. Hence Theorem 5.5 combined with Proposition 3.11 and [20, Thm. 4.2] yields the result. \square

Remark 5.8. If $F \in GL_N(\mathbb{C})$ satisfies $F\overline{F} \in \mathbb{R} \cdot \text{Id}$, one can consider the so-called *even part* of the discrete quantum group $\mathbb{F}O^+(F)$ which is the discrete quantum subgroup associated to the even representations of $O^+(F)$. By virtue of what precedes, these even parts have the Haagerup property and are weakly amenable with Cowling-Haagerup constant equal to 1. Through the monoidal equivalence of $O^+(F)$ with $SU_q(2)$, we see that the even part is monoidally equivalent to $SO_q(3)$ and that it is consequently isomorphic to the *quantum automorphism group* of some finite dimensional C^* -algebra (of dimension N^2) together with a (non-tracial) δ -form (see [34] for the definition and [18] for the details concerning monoidal equivalence). In the case $F = \text{Id}_N$, this quantum automorphism group can be completely identified.

Corollary 5.9. *Let N be an integer and let \mathbb{G}_N be the compact quantum automorphism group of $M_N(\mathbb{C})$ with respect to the canonical δ -trace (see [4] for details). Then $\Lambda_{cb}(\widehat{\mathbb{G}}_N) = 1$.*

Proof. According to [4, Corollary 4.1], $\widehat{\mathbb{G}}_N$ is the even part of the compact quantum $\mathbb{F}O_N^+$. \square

Remark 5.10. It was proved in [6] that $\widehat{\mathbb{G}}_N$ is isomorphic to the quantum permutation group $S_{N^2}^+$.

Remark 5.11. Let us also point out that the above results and the isomorphisms of [28, Thm 4.1] imply that the free bistochastic quantum groups B_N^+ and their symmetrized versions $(B_N^+)'$ (see e.g. [7] for the definitions) have the Haagerup property and are weakly amenable with Cowling-Haagerup constant equal to 1.

Taking a close look at the results of this section, one can see that our strategy to prove the Haagerup property for free orthogonal quantum groups can be adapted to the context of quantum automorphism groups of finite spaces. Let us end this paper with an outline of the argument (note that the Haagerup property has already been proved by M. Brannan in [11] when the δ -form is a trace).

Theorem 5.12. *Let B be a finite dimensional C^* -algebra together with a δ -form σ and let \mathbb{G} be the compact quantum automorphism group of (B, σ) . If $\widehat{\mathbb{G}}$ is not unimodular, then it has the Haagerup property.*

Proof. It was proved in [5] that the fusion rules of \mathbb{G} are that of $SO(3)$. Consequently, if we set $\pi_k(X) = \mu_{2k}(\sqrt{X})$ (which is a polynomial with integer coefficients), we can define a sequence of u.c.p multipliers $a(t) = \sum_k b_k(t)p_k$ by setting

$$b_k(t) = \frac{\pi_k(f_{it}(\tilde{\chi}_1))}{D_{2k}}.$$

These multipliers converge pointwise to the identity and we only have to check that they are in $C_0(\widehat{\mathbb{G}})$. This comes exactly from the same argument as in the proof of Theorem 5.5 (again using Lemma 5.4). \square

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UNIV. PARIS DIDEROT, SORBONNE PARIS CITÉ, UMR 7586, 175 RUE DU CHEVALERET,
75013, PARIS, FRANCE

E-mail address: freslon@math.jussieu.fr